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# On the Double Coset Membership Problem for Permutation Groups

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## Abstract

We show that the Double Coset Membership problem for permutation groups possesses perfect zero-knowledge proofs.

## 1 Introduction

### 1.1 Definition of the problem

Let  $S_m$  be a symmetric group of order  $m$ . We suppose that an element of  $S_m$ , a permutation of an  $m$ -element set, is encoded by a binary string of length  $n = \lceil \log_2 m! \rceil$ ,  $m(\log_2 m - O(1)) \leq n \leq m \log_2 m$ . Whenever we refer to a *permutation group*  $G$ , we mean that  $G$  is a subgroup of  $S_m$  for some  $m$ . Throughout the paper we assume that permutation groups are given by a list of their generators.

In this paper we address the following algorithmic problem considered first by Luks [21].

DCM (DOUBLE COSET MEMBERSHIP)

*Given:* two permutations  $\sigma$  and  $\tau$  and two permutation groups  $G$  and  $H$ , all of the same order.

*Recognize if:*  $\sigma \in G\tau H$ .

### 1.2 Current complexity status

For the background on computational complexity theory the reader is referred to [10].

DCM is in the class NP by the Babai-Szemerédi Reachability Theorem [5]. This theorem says that, given any set  $S$  of generators of a finite group  $G$  and any  $g \in G$ , there exists a sequence of elements  $u_1, \dots, u_l$  of  $G$  such that the following conditions are met.

1. Each  $u_i$  either belongs to  $S$  or is obtained by the inversion or the group operation from one or two previous elements of the sequence.
2.  $u_l = g$ .
3.  $l \leq (1 + \log_2 |G|)^2$ .

As  $\sigma \in G\tau H$  iff  $\tau^{-1}\sigma \in (\tau^{-1}G\tau)H$ , DCM admits the following reformulation.

DCM (An equivalent formulation)

*Given:* a permutation  $s$  and two permutation groups  $G$  and  $H$ , all of the same order.

*Recognize if:*  $s \in GH$ .

Consider two related problems, the first one easier and the second one harder than DCM.

MEMBERSHIP IN A PERMUTATION GROUP

*Given:* a permutation  $s$  and a permutation group  $G$  of the same order.

*Recognize if:*  $s \in G$ .

MEMBERSHIP IN A 3-FOLD GROUP PRODUCT

*Given:* a permutation  $s$  and three permutation groups  $G$ ,  $H$ , and  $K$ , all of the same order.

*Recognize if:*  $s \in GHK$ .

It is known that the former problem is solvable in polynomial time [25, 9] and that the latter problem is NP-complete [22]. There are evidences that the complexity of DCM is strictly in between. On the one hand, the problem of recognition if two given graphs are isomorphic is polynomial-time reducible to DCM [21], see also Proposition 3.2 below. DCM is therefore not expected to be solvable in polynomial time as long as the Graph Isomorphism problem is not solved in polynomial time (the currently best algorithm due to Luks and Zemplyachenko runs in time  $\exp(O(\sqrt{n} \log n))$  for graphs on  $n$  vertices, see [3]). On the other hand, DCM belongs to the complexity class coAM (see Subsection 2.1 for the definition). By [8], if NP is a subclass of coAM, then the polynomial-time hierarchy of complexity classes collapses to its second level, i.e.,  $\Sigma_2^P = \Pi_2^P$  (see [10]). As the latter consequence is widely considered unlikely, it is unlikely that DCM is NP-complete.

Like the membership in coAM, some other complexity-theoretic results known for Graph Isomorphism also generalize to DCM. Both the problems have *program checkers* [7], and both are *low* for the complexity class PP [20].

It is worth noting that several other group-theoretic problems are polynomial-time equivalent with DCM. We mention a few examples from the list of such problems compiled in [21, 19]: Given permutation groups  $G$ ,  $H$  and permutations  $\sigma$ ,  $\tau$ , (a) find generators for  $G \cap H$ ; (b) recognize if  $G\sigma$  and  $H\tau$  intersect; (c) if  $\sigma \in G$ , find the centralizer of  $\sigma$  in  $G$ ; (d) if  $\sigma, \tau \in G$ , recognize if the centralizer of  $\tau$  in  $S_m$  intersects  $G\sigma$ . In [7] it is shown that DCM is equivalent with the problem, given  $s \in GH$ , to find a factorization  $s = gh$  with  $g \in G$  and  $h \in H$ .

### 1.3 Our result

A natural question to ask about an NP problem whose polynomial-time solvability and NP-completeness are unknown is if it possesses a perfect or a statistical zero-knowledge interactive proof system. Informally speaking, a zero-knowledge proof system for a recognition problem of a language  $L$  is a protocol for two parties, the prover and the verifier, that allows the prover to convince the verifier that a given input belongs to  $L$ , with high confidence but without communicating the verifier any information (the rigorous definitions are in Subsection 2.1).

The concept of a zero-knowledge proof has notable applications in designing cryptographic protocols and in estimating the computational complexity of a language recognition problem. Namely, by [1] the class PZK of languages having perfect zero-knowledge proof systems is a subclass of coAM. Thus, the existence of a perfect zero-knowledge proof of the membership in  $L$  not only has a cryptographic meaning but also implies that  $L$  is in coAM and hence cannot be NP-complete unless the polynomial-time hierarchy collapses.

For the Graph Isomorphism problem, its membership in coAM was proven directly in [24] and its membership in PZK was proven in [14]. For DCM, the proof of its membership in coAM given in [4] is direct. In the present paper we prove that DCM is also in PZK. We therefore extend the list of problems in PZK that currently includes Graph Isomorphism [14], Quadratic Residuosity [16], a problem equivalent to Discrete Logarithm [13], and approximate versions of the Shortest Vector and Closest Vector problems for integer lattices [11].

## 2 Background on zero-knowledge proofs

### 2.1 Definitions

We denote the *length* of a binary word  $w$  by  $|w|$ . We consider *languages* over the binary alphabet which are subsets of  $\{0,1\}^*$ . The *complement* of  $L$  is the language  $\bar{L} = \{0,1\}^* \setminus L$ . Note that the DCM problem can be represented as a recognition problem for the language  $L = \{(s, G, H) : s \in GH\}$ , where  $(s, G, H)$  is a suitable binary encoding of the triplet consisting of a permutation  $s$  and the lists of generators for permutation groups  $G$  and  $H$ .

We use the standard computational model of a deterministic *Turing machine*, abbreviated further on as TM. We assume that a TM has three tapes, namely, the input tape, the output tape, and the work tape where all computations are performed.

A *probabilistic* TM, abbreviated further on as PTM, in addition has the fourth tape containing a potentially infinite random binary string. Assuming that a PTM halts on input  $w$  and random string  $r$ , we denote its running time by  $t(w, r)$ . A PTM is *polynomial-time* if  $t(w, r)$  is bounded by a polynomial in  $|w|$  for all  $w$  and  $r$ . Assuming that a PTM halts on  $w$  for almost all  $r$ , the function  $t(w, r)$  for a fixed  $w$  can be considered as a random variable on the probability space  $\{0,1\}^{\mathbb{N}}$  of all

random strings. A PTM is *expected polynomial-time* on  $L \subseteq \{0, 1\}^*$  if for all  $w \in L$  the expectation of  $t(w, r)$  is bounded by a polynomial in  $|w|$ .

An *interactive proof system*  $\langle V, P \rangle$ , further on abbreviated as IPS, consists of two PTMs, a polynomial-time  $V$  called *the verifier* and a computationally unlimited  $P$  called *the prover*. The input tape is common for the verifier and the prover. The verifier and the prover also share a communication tape which allows message exchange between them. The system works as follows. First both the machines  $V$  and  $P$  are given an input  $w$  and each of them is given an individual random string,  $r_V$  for  $V$  and  $r_P$  for  $P$ . Then  $P$  and  $V$  alternately write messages to one another in the communication tape.  $V$  computes its  $i$ -th message  $a_i$  to  $P$  based on the input  $w$ , the random string  $r_V$ , and all previous messages from  $P$  to  $V$ .  $P$  computes its  $i$ -th message  $b_i$  to  $V$  based on the input  $w$ , the random string  $r_P$ , and all previous messages from  $V$  to  $P$ . After a number of message exchanges  $V$  terminates interaction and computes an output based on  $w$ ,  $r_V$ , and all  $b_i$ . The output is denoted by  $\langle V, P \rangle(w)$ . Note that, for a fixed  $w$ ,  $\langle V, P \rangle(w)$  is a random variable depending on both random strings  $r_V$  and  $r_P$ .

Let  $\epsilon(n)$  be a function of a natural argument taking on positive real values. We call  $\epsilon(n)$  *negligible* if  $\epsilon(n) < n^{-c}$  for every  $c$  and all  $n$  starting from some  $n_0(c)$ . For example, an *exponentially small* function  $\epsilon(n) = d^{-n}$ , where  $d > 1$ , is negligible.

We say that  $\langle V, P \rangle$  is an *IPS for a language  $L$  with error  $\epsilon(n)$*  if the following two conditions are fulfilled.

*Completeness.* If  $w \in L$ , then  $\langle V, P \rangle(w) = 1$  with probability at least  $1 - \epsilon(|w|)$ .

*Soundness.* If  $w \notin L$ , then, for an arbitrary interacting PTM  $P^*$ ,  $\langle V, P^* \rangle(w) = 1$  with probability at most  $\epsilon(|w|)$ .

We will call any prover  $P^*$  interacting with  $P$  on input  $w \notin L$  *cheating*. If in the completeness condition we have  $\langle V, P \rangle(w) = 1$  with probability 1, we say that  $\langle V, P \rangle$  has *one-sided error*  $\epsilon(n)$ .

We say that  $\langle V, P \rangle$  is an *IPS for a language  $L$*  if  $\langle V, P \rangle$  is an IPS for  $L$  with negligible error.

An IPS is *public-coin* if the concatenation  $a_1 \dots a_k$  of the verifier's messages is a prefix of his random string  $r_V$ . A *round* is sending one message from the verifier to the prover or from the prover to the verifier. The class AM consists of those languages having IPSs with error  $1/3$  and with number of rounds bounded by a constant for all inputs. A language  $L$  belongs to the class coAM iff its complement  $\bar{L}$  belongs to AM.

Given an IPS  $\langle V, P \rangle$  and an input  $w$ , let  $\text{view}_{V,P}(w) = (r'_V, a_1, b_1, \dots, a_k, b_k)$  where  $r'_V$  is a part of  $r_V$  scanned by  $V$  during work on  $w$  and  $a_1, b_1, \dots, a_k, b_k$  are all messages from  $P$  to  $V$  and from  $V$  to  $P$  ( $a_1$  may be empty if the first message is sent by  $P$ ). Note that the verifier's messages  $a_1, \dots, a_k$  could be excluded because they are efficiently computable from the other components. For a fixed  $w$ ,  $\text{view}_{V,P}(w)$  is a random variable depending on  $r_V$  and  $r_P$ .

An IPS  $\langle V, P \rangle$  is *perfect zero-knowledge on  $L$*  if for every interacting polynomial-time PTM  $V^*$  there is a PTM  $M_{V^*}$ , called a *simulator*, that on every input  $w \in L$  runs in expected polynomial time and produces output  $M_{V^*}(w)$  which, if considered

as a random variable depending on a random string of  $M_{V^*}$ , is distributed identically with  $\text{view}_{V^*,P}(w)$ . The latter condition means that

$$\mathbf{P}[M_{V^*}(w) = z] = \mathbf{P}[\text{view}_{V^*,P}(w) = z] \text{ for all } z.$$

If only a weaker condition that

$$\sum_z |\mathbf{P}[M_{V^*}(w) = z] - \mathbf{P}[\text{view}_{V^*,P}(w) = z]| \text{ is negligible}$$

is true, we call  $\langle V, P \rangle$  *statistical zero-knowledge*. These notions formalize the claim that the verifier gets no information during interaction with the prover: Everything that the verifier gets he can get without the prover by running the simulator.

According to the definition the verifier learns nothing even if he deviates from the original program and follows an arbitrary probabilistic polynomial-time program  $V^*$ . We will call the verifier  $V$  *honest* and all other verifiers  $V^*$  *cheating*. If the existence of a simulator is claimed only for the honest verifier, we call such a proof system *honest-verifier perfect* (or *statistical*) *zero-knowledge*.

The class of languages  $L$  having IPSs that are perfect (resp. statistical) zero-knowledge on  $L$  is denoted by PZK (resp. SZK). Recall that the error here is supposed negligible.

The  $k(n)$ -fold *sequential composition* of an IPS  $\langle V, P \rangle$  is the IPS  $\langle V', P' \rangle$  in which  $V'$  and  $P'$  on input  $w$  execute the programs of  $V$  and  $P$  sequentially  $k(|w|)$  times, each time with independent choice of random strings  $r_V$  and  $r_P$ . At the end of interaction  $V'$  outputs 1 iff  $\langle V, P \rangle(w) = 1$  in all  $k(|w|)$  executions. The initial system  $\langle V, P \rangle$  is called *atomic*.

In the  $k(n)$ -fold *parallel composition*  $\langle V'', P'' \rangle$  of  $\langle V, P \rangle$ , the program of  $\langle V, P \rangle$  is executed  $k(|w|)$  times in parallel, that is, in each round all  $k(|w|)$  versions of a message are sent from one machine to another at once as a long single message. In every parallel execution  $V''$  and  $P''$  use independent copies of  $r_V$  and  $r_P$ . At the end of interaction  $V''$  outputs 1 iff  $\langle V, P \rangle(w) = 1$  in all  $k(|w|)$  executions.

## 2.2 Known results on zero-knowledge proofs

We first notice a simple property of sequential composition of IPSs.

**Proposition 2.1** *If  $\langle V, P \rangle$  is an IPS for a language  $L$  with one-sided constant error  $\epsilon$ , then the  $k(n)$ -fold sequential composition of  $\langle V, P \rangle$  is an IPS for  $L$  with one-sided error  $\epsilon^{k(n)}$ .*

Parallel composition obviously preserves the number of rounds, the public-coin property, and the property of error to be one-sided. It is not hard to prove that  $k$ -fold parallel composition reduces the one-sided error  $\epsilon$  to  $\epsilon^k$ . It is also not hard to prove that parallel composition preserves perfect and statistical zero-knowledge for the honest verifier. These observations are summarized in the next proposition.

**Proposition 2.2** *Assume that  $\langle V, P \rangle$  is a honest-verifier perfect zero-knowledge public-coin IPS for a language  $L$  that on all inputs works in a constant  $c$  rounds with one-sided constant error  $\epsilon$ . Then  $k(n)$ -fold parallel composition of  $\langle V, P \rangle$  is a honest-verifier perfect zero-knowledge IPS for  $L$  that works in  $c$  rounds with error  $\epsilon^{k(n)}$ .*

We also refer to the following deep results in the theory of zero-knowledge proofs.

**Proposition 2.3 (Aiello-Håstad [1])**  $SZK \subseteq coAM$ .

**Proposition 2.4 (Okamoto [23])**

1. *Every honest-verifier statistical zero-knowledge IPS for a language  $L$  can be transformed in an honest-verifier statistical zero-knowledge public-coin IPS for  $L$ .*
2. *If  $L$  has an honest-verifier statistical zero-knowledge public-coin IPS, then  $\bar{L}$  has a honest-verifier statistical zero-knowledge constant-round IPS.*

Note that the item 2 of Proposition 2.4 strengthens Proposition 2.3 because by [17] every IPS can be made public-coin at cost of decreasing the number of rounds in 2.

**Proposition 2.5 (Goldreich-Sahai-Vadhan [15])** *Every honest-verifier statistical zero-knowledge public-coin IPS for a language  $L$  can be transformed in a general statistical zero-knowledge public-coin IPS for  $L$ . If the error of the initial IPS is one-sided, so is the error of the resulting IPS.*

Note that, to achieve the negligible error, the transformation of Proposition 2.5 makes the number of rounds increasing with the input size increasing, even if the initial IPS is constant-round. A transformation preserving the constant number of rounds is known only under an unproven assumption about the hardness of the Discrete Logarithm problem (the formal statement of the assumption can be found in [6]).

**Proposition 2.6 (Bellare-Micali-Ostrovsky [6])** *Suppose that a language  $L$  has an honest-verifier statistical zero-knowledge IPS that on every input  $w$  works in  $c(|w|)$  rounds with error at most  $1/3$ . Then, under the assumption on the hardness of Discrete Logarithm,  $L$  has a general statistical zero-knowledge IPS that on input  $w$  works in  $O(c(|w|))$  rounds with exponentially small error.*

### 3 Background on permutation groups

Given a finite set  $X$ , by a *random element* of  $X$  we mean a random variable uniformly distributed over  $X$ .

**Proposition 3.1 (Sims [25, 9])**

1. *There is a polynomial-time algorithm for recognizing the MEMBERSHIP IN A PERMUTATION GROUP.*
2. *There is a probabilistic polynomial-time algorithm that, given a list of generators for a permutation group  $G$ , outputs a random element of  $G$ .*

The DCM problem is at least as hard as testing isomorphism of two given graphs.

**Proposition 3.2 (Luks [21], Hoffmann [18])** *The Graph Isomorphism problem is polynomial-time reducible to DCM.*

We include a proof for the sake of completeness.

*Proof.* Consider two graphs of order  $n$  with adjacency matrices  $A = (a_{ij})$  and  $B = (b_{ij})$ . Let  $S_1 = \{(i, j) : a_{ij} = 1\}$  and  $S_2 = \{(i, j) : b_{ij} = 1\}$ .

Let  $G$  be the group of permutations of the square  $\{1, \dots, n\}^2$  generated by simultaneous transpositions of  $i$ -th and  $j$ -th rows and  $i$ -th and  $j$ -th columns for all  $1 \leq i < j \leq n$ . The graphs are isomorphic iff  $G$  contains a permutation  $\sigma$  such that  $\sigma(S_1) = S_2$ .

Let  $H$  be the group of permutations  $\tau$  such that  $\tau(S_1) = S_1$  and  $s$  be an arbitrary permutation such that  $s(S_1) = S_2$ . As easily seen, a permutation  $\sigma$  as above exists iff  $s \in GH$ .  $\square$

Note that the reduction described allows one to transform any zero-knowledge proof system for DCM in a zero-knowledge proof system for Graph Isomorphism.

## 4 Zero-knowledge proofs for DCM

**Theorem 4.1** *The DCM problem has an honest-verifier perfect zero-knowledge three-round public-coin IPS with one-sided error  $1/2$ .*

*Proof.* On input  $(s, G, H)$  such that  $s \in GH$  the IPS  $\langle V, P \rangle$  proceeds as follows.

*1st round.*

$P$  generates random elements  $g \in G$  and  $h \in H$ , computes  $t = gsh$ , and sends  $t$  to  $V$ .  $V$  checks if  $t$  is a permutation of the given order and if not (this is possible in the case of a cheating prover) halts and outputs 1.

*2nd round.*

$V$  chooses a random bit  $b \in \{0, 1\}$  and sends it to  $P$ .

*3rd round.*

*Case  $b = 0$ .*  $P$  sends  $V$  permutations  $g$  and  $h$ .  $V$  checks if  $g \in G$ ,  $h \in H$ , and  $t = gsh$ .

*Case  $b \neq 0$*  (this includes the possibility of a message  $b \notin \{0, 1\}$  produced by a cheating verifier).  $P$  decomposes  $s$  into the product  $s = g_0 h_0$  with  $g_0 \in G$  and  $h_0 \in H$ , computes  $g_1 = g g_0$  and  $h_1 = h_0 h$ , and sends  $g_1$  and  $h_1$  to  $V$ .  $V$  checks if  $g_1 \in G$ ,  $h_1 \in H$ , and  $t = g_1 h_1$ .

$V$  halts and outputs 1 if the conditions are checked successfully and 0 otherwise.

This IPS is obviously public-coin. We need to check that this is indeed an IPS for DCM with one-sided error  $1/2$  and, moreover, that this is a honest-verifier perfect zero-knowledge IPS.

*Completeness.* If  $s \in GH$ , then it is clear that  $V$  outputs 1 with probability 1.

*Soundness.* Assume that  $s \notin GH$  and consider an arbitrary cheating prover  $P^*$ . Observe that if both  $t = gsh$ ,  $g \in G$ ,  $h \in H$  and  $t = g_1h_1$ ,  $g_1 \in G$ ,  $h_1 \in H$ , then  $s \in GH$ . It follows that, for at least one value of  $b$ ,  $V$  outputs 0 and therefore  $V$  outputs 1 with probability at most  $1/2$ .

*Zero-knowledge.* Assume that  $s \in GH$ . During interaction with  $P$ ,  $V$  sees  $\text{view}_{V,P}(s, G, H) = (b, t, b, g', h')$  where  $g'$  and  $h'$  are received by  $V$  in the 3rd round. If  $b = 0$ , then  $t = gsh$ ,  $g' = g$ , and  $h' = h$ . If  $b = 1$ , then  $t = g'h'$ ,  $g' = gg_0$ , and  $h' = h_0h$ . In both the cases  $g'$  and  $h'$  are random elements of  $G$  and  $H$  respectively. The random variable  $\text{view}_{V,P}(s, G, H)$  can be therefore generated by the following simulator: Generate a random bit  $b$  and random elements  $g' \in G$  and  $h' \in H$ ; If  $b = 0$ , set  $t = g'sh'$ ; If  $b = 1$ , set  $t = g'h'$ .  $\square$

**Corollary 4.1** *The DCM problem has an honest-verifier perfect zero-knowledge three-round public-coin IPS with one-sided error  $2^{-n}$ .*

*Proof.* By Proposition 2.2 the  $n$ -fold parallel composition of the IPS from Theorem 4.1 reduces the error to  $2^{-n}$  and preserves the properties of the atomic system.  $\square$

Let DOUBLE COSET NON-MEMBERSHIP, abbreviated as DCNM, be the problem opposite to DCM, that is, given a permutation  $s$  and two permutation groups  $G$  and  $H$ , to recognize if  $s \notin GH$ . The DCNM problem is clearly polynomial-time equivalent with recognition of the set-theoretic complement of DCM, where the latter is encoded as a language in the binary alphabet.

**Corollary 4.2** *DCNM has an honest-verifier statistical zero-knowledge constant-round IPS.*

*Proof.* The corollary follows from Corollary 4.1 by Proposition 2.4.

We also give an alternative direct proof of this claim describing an honest-verifier perfect zero-knowledge two-round IPS  $\langle V, P \rangle$  for DCNM with one-sided error  $1/2$ . This system, for the case of permutation groups, generalizes the IPS suggested in [2] for the problem of testing the membership in a finite group given by a list of generators and an oracle access to the group operation.

On input  $(s, G, H)$  such that  $s \notin GH$  the system works as follows.

*1st round.*

$V$  chooses a random bit  $b$  to be the first bit of a random string  $r_V$  and, based on the subsequent bits of  $r_V$ , generates random elements  $g \in G$  and  $h \in H$ . If  $b = 0$ ,  $V$  computes  $t = gh$ ; If  $b = 1$ ,  $V$  computes  $t = gsh$ . Then  $V$  sends  $t$  to  $P$ .

*2nd round.*

$P$  recognizes if  $t \in GH$ . If so,  $P$  sets  $a = 0$ ; If not,  $P$  sets  $a = 1$ . Then  $P$  sends  $a$  to  $V$ .



$V$  checks if  $a = b$  and halts. If the equality is true,  $V$  outputs 1; Otherwise  $V$  outputs 0.

*Completeness.* Assume that  $s \notin GH$ . In the first round,  $t \in GH$  if  $b = 0$  and  $t \notin GH$  if  $b = 1$ . Therefore  $V$  outputs 1 with probability 1.

*Soundness.* Assume that  $s \in GH$ . Then  $t \in GH$  regardless of the value of  $b$ . Moreover,  $t$  is the product of random elements of  $G$  and  $H$  and, as a random variable, is independent of the random variable  $b$ . It follows that in the second round a message from the cheating prover  $P^*$  to  $V$ , which is a function of  $s, G, H, r_{P^*}$ , and  $t$ , is equal to  $b$  with probability at most  $1/2$ . Hence  $\langle V, P^* \rangle(s, G, H) = 1$  with probability at most  $1/2$ .

*Zero-knowledge.* Assume that  $s \notin GH$ . During interaction with  $P$ ,  $V$  sees  $\text{view}_{V,P}(s, G, H) = (r'_V, t, a)$ , where  $a$  equals the first bit  $b$  of  $r'_V$ . The simulator therefore just generates a random string  $r_V$ , extracts the first bit  $b$  from it, sets  $a = b$ , based on the remaining bits of  $r_V$  computes  $g$  and  $h$ , based on  $b, g$ , and  $h$  computes  $t$ , and sets  $r'_V$  to be the prefix of  $r_V$  that was actually used for these purposes.  $\square$

**Corollary 4.3** *DCM is in SZK. Moreover, DCM has a statistical zero-knowledge public-coin IPS with one-sided error.*

*Proof.* Apply the transformation from Proposition 2.5 to the IPS from Corollary 4.1.

Note that another proof of the membership of DCM in SZK can be given by applying Propositions 2.4 and 2.5 to the IPS in the alternative proof of Corollary 4.2.  $\square$

**Corollary 4.4 (Babai-Moran [4])** *DCM is in coAM. Therefore DCM is not NP-complete unless the polynomial-time hierarchy collapses at the second level.*

*Proof.* This is an immediate consequence of Corollary 4.2 or a consequence of Corollary 4.3 based on Proposition 2.3.  $\square$

**Corollary 4.5** *Under the assumption on the hardness of Discrete Logarithm, DCM has a constant-round statistical zero-knowledge IPS with exponentially small error.*

*Proof.* The corollary follows from Theorem 4.1 by Proposition 2.6.  $\square$

**Theorem 4.2** *The  $n$ -fold sequential composition of the IPS in Theorem 4.1 is a perfect zero-knowledge public-coin IPS for DCM with exponentially small error. Hence DCM is in PZK.*

*Proof.* Denote the composed IPS by  $\langle V, P \rangle$ . As the atomic system is public-coin, so is  $\langle V, P \rangle$ . By Proposition 2.1  $\langle V, P \rangle$  is an IPS for DCM with one-sided error  $2^{-n}$ . We have to prove that  $\langle V, P \rangle$  is perfect zero-knowledge.

For each verifier  $V^*$  interacting with  $P$  we describe a probabilistic expected polynomial-time simulator  $M_{V^*}$ . The  $M_{V^*}$  uses the program of  $V^*$  as a subroutine. Assume that the running time of  $V^*$  is bounded by a polynomial  $q(n)$  in the input size. On input  $w$ ,  $M_{V^*}$  will run the program of  $V^*$  on input  $w$  with random string  $r$ , where  $r$  is the prefix of  $M_{V^*}$ 's random string of length  $q(|w|)$ . In all other cases  $M_{V^*}$  will use the remaining part of its random string.

Work of  $M_{V^*}$  on input  $w = (s, G, H)$  consists of  $|w|$  stages, where a stage corresponds to an iteration of the atomic system.

*Stage  $i$ .*

$M_{V^*}$  chooses random elements  $g_i \in G$  and  $h_i \in H$  and a random bit  $a \in \{0, 1\}$ . If  $a = 0$ ,  $M_{V^*}$  computes  $t_i = g_i s h_i$ ; If  $a = 1$ , it computes  $t_i = g_i h_i$ . Then  $M_{V^*}$  computes  $b_i = V^*(w, r, t_1, g_1, h_1, \dots, t_{i-1}, g_{i-1}, h_{i-1}, t_i)$ , the message that  $V^*(w, r)$  sends  $P$  in the  $i$ -th sequential iteration of the atomic system after receiving  $P$ 's message  $t_i$  and under the condition that in the preceding iterations  $P$ 's messages were  $t_1, g_1, h_1, \dots, t_{i-1}, g_{i-1}, h_{i-1}$ . If  $b_i$  and  $a$  are simultaneously equal to or different from 0, then  $M_{V^*}$  puts  $v_i = (t_i, b_i, g_i, h_i)$  and proceeds to the  $(i + 1)$ -th stage. If exactly one of  $b_i$  and  $a$  is equal to 0, then  $M_{V^*}$  restarts the same  $i$ -th stage with new independent choice of  $a, g_i, h_i$ .

After all stages are completed,  $M_{V^*}$  halts and outputs  $(r', v_1, \dots, v_{|w|})$ , where  $r'$  is the prefix of  $r$  actually used by  $V^*$  during interaction on input  $w$  with the prover sending the messages  $t_1, g_1, h_1, \dots, t_{|w|}, g_{|w|}, h_{|w|}$ . Notice that it might happen that in unsuccessful attempts to pass some stage  $V^*$  used a prefix of  $r$  longer than  $r'$ .

We first check that  $M_{V^*}$  terminates in expected polynomial time whenever  $s \in GH$ . Since  $V^*$  is polynomial-time, one attempt to pass Stage  $i$ ,  $i \leq |w|$ , takes time bounded by a polynomial in  $|w|$ . Recall that  $M_{V^*}$  is programmed so that  $a$  and  $r$  are independent. Furthermore,  $a$  and  $t_i$  are independent. Indeed, if  $a = 1$ , then  $t_i = g_i h_i$  is the product of random elements of  $G$  and  $H$ . If  $a = 0$ , then  $t_i = (g_i g_0)(h_0 h_i)$  is such a product as well. Here  $g_0 \in G$  and  $h_0 \in H$  are elements of an arbitrary decomposition  $s = g_0 h_0$ . It follows that  $a$  and  $b_i$  are independent and therefore an execution of the stage is successful with probability  $1/2$ . We conclude that on average each stage consists of 2 executions. Thus, on average  $M_{V^*}$  makes  $2|w|$  polynomial-time executions and this takes expected polynomial time.

We finally need to check that, whenever  $s \in GH$ , the output  $M_{V^*}(w)$  is distributed identically with  $\text{view}_{V^*, P}(w)$ . Notice that both the random variables depend on  $V^*$ 's random string  $r$ . It therefore suffices to show that the distributions are identical when conditioned on an arbitrary fixed  $r$ . For  $0 \leq i \leq |w|$ , let  $D_M^i(w, r)$  denote the probability distribution of  $(r', v_1, \dots, v_i)$  conditioned on  $r$ , and  $D_{V^*, P}^i(w, r)$  denote the distribution of the part of  $\text{view}_{V^*, P}(w)$  formed up to the  $i$ -th sequential iteration. With this notation, we have to prove that  $D_M^{|w|}(w, r) = D_{V^*, P}^{|w|}(w, r)$ . Using the induction on  $i$ , we prove that  $D_M^i(w, r) = D_{V^*, P}^i(w, r)$  for every  $0 \leq i \leq |w|$ .

The base case of  $i = 0$  is trivial. Let  $i \geq 1$  and assume that

$$\mathbf{P} \left[ D_M^{i-1}(w, r) = u_{i-1} \right] = \mathbf{P} \left[ D_{V^*, P}^{i-1}(w, r) = u_{i-1} \right] \quad (1)$$

for every value  $u_{i-1}$ . Given  $u_{i-1}$ , assume now that both  $D_M^{i-1}(w, r) = u_{i-1}$  and  $D_{V^*,P}^{i-1}(w, r) = u_{i-1}$ , and under these conditions consider how the  $i$ -th components  $v_i = (t_i, b_i, g_i, h_i)$  are distributed in  $u_i = u_{i-1}v_i$  according to  $D_M^i(w, r)$  and  $D_{V^*,P}^i(w, r)$ . We will show that

$$\begin{aligned} \mathbf{P} \left[ D_M^i(w, r) = u_{i-1}v_i \mid D_M^{i-1}(w, r) = u_{i-1} \right] \\ = \mathbf{P} \left[ D_{V^*,P}^i(w, r) = u_{i-1}v_i \mid D_{V^*,P}^{i-1}(w, r) = u_{i-1} \right] \end{aligned} \quad (2)$$

for every value  $v_i$ . Together with (1) this will imply the identity of  $D_M^i(w, r)$  and  $D_{V^*,P}^i(w, r)$ .

To prove (2), we will show that according to the both conditional distributions  $v_i$  is uniformly distributed on the set

$$\begin{aligned} S = \left\{ (t, b, g, h) \ : \ t \in GH, \ b = V^*(w, r, u_{i-1}, t), \ g \in G, \ h \in H, \right. \\ \left. t = gsh \text{ if } b = 0 \text{ and } t = gh \text{ if } b \neq 0 \right\}. \end{aligned}$$

Given  $t$  and  $s$ , define sets  $R(t) = \{(g, h) : g \in G, h \in H, gh = t\}$  and  $R_s(t) = \{(g, h) : g \in G, h \in H, gsh = t\}$ . The first claim of the following lemma appeared in [18].

**Lemma 4.1** *Let  $k = |G \cap H|$ . Assume that  $s = g_0h_0$  with  $g_0 \in G$  and  $h_0 \in H$ . Then the following statements are true.*

1. Every  $t \in GH$  has  $k$  representations  $t = gh$  with  $g \in G$  and  $h \in H$ , i.e.,  $|R(t)| = k$ . If  $t = g_1h_1$ , then all other representations are

$$t = (g_1f)(f^{-1}h_1), \quad (3)$$

where  $f$  ranges over group  $G \cap H$ .

2. For every  $t$ , the mapping  $\alpha(g, h) = (gg_0, h_0h)$  is one-to-one from  $R_s(t)$  to  $R(t)$ .
3. Every  $t \in GH$  has  $k$  representations  $t = gsh$  with  $g \in G$  and  $h \in H$ , i.e.,  $|R_s(t)| = k$ .
4. If  $\phi : G \times H \rightarrow GH$  is defined by  $\phi(g, h) = gh$ , then  $|\phi^{-1}(t)| = k$  for every  $t \in GH$ .
5. If  $\psi : G \times H \rightarrow GH$  is defined by  $\psi(g, h) = gsh$ , then  $|\psi^{-1}(t)| = k$  for every  $t \in GH$ .
6. If  $t = gh$  is the product of uniformly distributed random elements  $g \in G$  and  $h \in H$ , then  $t$  is uniformly distributed on  $GH$ .
7. If a uniformly distributed random pair  $(g, h) \in G \times H$  is conditioned on  $gh = t$  for an arbitrary fixed  $t \in GH$ , then  $(g, h)$  is uniformly distributed on  $R(t)$ .

8. If  $t = gsh$  and  $g \in G$  and  $h \in H$  are uniformly distributed random elements, then  $t$  is uniformly distributed on  $GH$ .
9. If a uniformly distributed random pair  $(g, h) \in G \times H$  is conditioned on  $gsh = t$  for an arbitrary fixed  $t \in GH$ , then  $(g, h)$  is uniformly distributed on  $R_s(t)$ .

*Proof.* We first prove Item 1. Let  $e$  denote the identity permutation. Clearly that we have at least  $k$  representations of the form (3). On the other hand, every representation  $t = gh$  is of this form. Indeed, we have  $(g^{-1}g_1)(h_1h^{-1}) = e$  and hence both  $g^{-1}g_1$  and  $h_1h^{-1}$  are simultaneously in  $G$  and in  $H$ .

To prove Item 2, observe that  $\alpha$  is indeed from  $R_s(t)$  to  $R(t)$ . The map  $\alpha'(g, h) = (gg_0^{-1}, h_0^{-1}h)$  is easily seen to be from  $R(t)$  to  $R_s(t)$  and inverse to  $\alpha$ .

Items 1 and 2 imply Item 3, Item 3 implies Item 5, and Item 5 implies Item 8. Item 1 implies Item 4, and Item 4 implies Item 6. Items 7 and 9 are true by the definition of  $R(t)$  and  $R_s(t)$ .  $\square$

The distribution  $D_{V^*,P}^i(w, r)$  conditioned on  $D_{V^*,P}^{i-1}(w, r) = u_{i-1}$  is samplable as follows. Choose random elements  $g \in G$  and  $h \in H$ . Compute  $t_i = gsh$  and  $b_i = V^*(w, r, u_{i-1}, t_i)$ . If  $b_i = 0$ , set  $g_i = g$  and  $h_i = h$ , otherwise set  $g_i = gg_0$  and  $h_i = h_0h$ . Clearly, this distribution of  $(t_i, b_i, g_i, h_i)$  is over  $S$ .

By Item 8 of Lemma 4.1,  $t_i$  is uniformly distributed on  $GH$ . If  $b_i = 0$ , then by Item 9 of Lemma 4.1, for every fixed  $t_i$ , the pair  $(g_i, h_i)$  is uniformly distributed on  $R_s(t)$ . If  $b_i \neq 0$ , then by Item 2 of Lemma 4.1, for every fixed  $t_i$ , the pair  $(g_i, h_i)$  is uniformly distributed on  $R(t)$ . It follows that  $D_{V^*,P}^i(w, r)$  conditioned on  $D_{V^*,P}^{i-1}(w, r) = u_{i-1}$  is uniform on  $S$ .

Consider now the sampling procedure for the distribution  $D_M^i(w, r)$  conditioned on  $D_M^{i-1}(w, r) = u_{i-1}$  as in the description of the simulator  $M_{V^*}$ . Under the condition that  $a = 0$ , by Items 8 and 9 of Lemma 4.1,  $t_i$  is distributed uniformly over  $GH$  and for every fixed value of  $t_i$ , the pair  $(g_i, h_i)$  is uniformly distributed over  $R_s(t)$ . Under the condition that  $a = 1$ , by Items 6 and 7 of Lemma 4.1,  $t_i$  is distributed uniformly over  $GH$  and for every fixed value of  $t_i$ , the pair  $(g_i, h_i)$  is uniformly distributed over  $R(t)$ . This leads to an equivalent sampling procedure: Choose a random  $t_i \in GH$ , compute  $b_i = V^*(w, r, u_{i-1}, t_i)$ ; If  $b_i = 0$ , choose a random pair  $(g_i, h_i)$  in  $R_s(t_i)$ , otherwise in  $R(t)$ . It follows that  $D_M^i(w, r)$  conditioned on  $D_M^{i-1}(w, r) = u_{i-1}$  is uniform on  $S$ .  $\square$

**Remark 4.1** The simulator in the proof of Theorem 4.2 is *black-box*, that is, for each  $V^*$  it follows the same program that uses the strategy of  $V^*$  as a subroutine. It should be noted that by [12] the parallel composition of the IPS in Theorem 4.1 is *not* zero-knowledge with black-box simulator unless DCM is decidable in probabilistic polynomial time.

## 5 Future work

A natural question arises if our results can be extended to *matrix groups* over finite fields. One of the reasons why this case is more complicated is that, unlike permuta-

tion groups, no efficient test of membership for matrix groups is known. We intend to tackle this question in a subsequent paper.

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